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Nonlinear Flexural Vibrations of a Rotating Myklestad Beam

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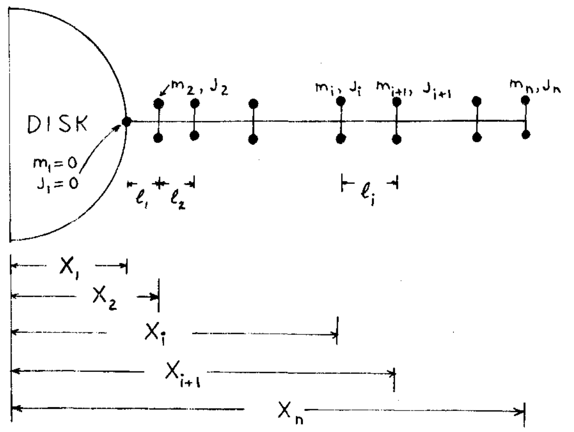

Fig. 1 Myklestad beam model of blade.

Introduction

THE equations of motion of a rotating cantilevered blade vibrating in a plane other than the one perpendicular to the plane of rotation are nonlinear because of the couplings due to Coriolis forces. Carnegie[1] used a variational approach to derive the partial differential equations of motion of a pretwisted cantilevered blade mounted on the periphery of a rotating disk. His equations include the effects of rotatory inertia, torsion, bending and Coriolis accelerations, but are complicated and only solvable for very special cases. Rao and Carnegie[2] considered a uniform, untwisted, rotating cantilevered blade vibrating in its plane of rotation and simplified the problem considerably by ignoring the shear deformation and rotatory inertia effects. The fundamental nonlinear mode shape was obtained by the Ritz method. The purpose of this Note is to generate the nonlinear modes of a nonuniform rotating blade performing inplane vibrations by resorting to a Myklestad beam which is referred to here as a discrete model described in Ref. 3.

Analysis

Consider a nonuniform, untwisted, radial, cantilevered blade with a vertical plane of symmetry, mounted on the periphery of a disk rotating with a uniform angular velocity Ω about its polar axis. The Myklestad beam model of the blade, as shown in Fig. 1, has its inertia properties concentrated at n discrete mass stations along its axis, the beam sections between consecutive stations being massless but possessing bending and shear flexibilities. Each station i has a mass m_i and a bending inertia J_i located at a distance x_i from the axis of rotation. The section length from station i to station $i+1$ is l_i. The beam axis is assumed inextensional and vibration is restricted to the plane of rotation. Amplitudes are taken to be moderately large and therefore any deviation from the linear modes of the system would be small.

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Using a two-mode approximation, we can write down expressions for the lateral displacement y_i and the beam slope α_i at the ith mass station in terms of the normal modes of the linear rotating beam as follows:

$$y_i = \sum_{j=1}^{2} A_i^{(j)} q_j(t), \qquad \alpha_i = \sum_{j=1}^{2} C_i^{(j)} q_j(t) \tag{1}$$

where $A_i^{(j)}$ is the ith component of the jth normal mode $\{A\}^{(j)}$, $C_i^{(j)}$ is the ith component of the associated slope vector $\{C\}^{(j)}$, and the $q_j(t)$ are normal coordinates. Including the contribution due to the centrifugal force field, the potential energy of the rotating Myklestad beam can now be written as[4]

$$V = (1/2) \sum_{j=1}^{2} \omega_{(j)}^2 M_{(j)} q_j^2 \tag{2}$$

where $\omega_{(j)}$ is the jth natural frequency of the linear rotating beam, and

$$M_{(j)} = \sum_{i=1}^{n} (m_i A_i^{(j)2} + J_i C_i^{(j)2}) \tag{3}$$

is the associated generalized mass.[4]

The potential energy expression given by Eq. (2) includes the effect of the centrifugal force field that acts upon the blade, which must now be considered nonrotating for the purpose of computing its kinetic energy. This kinetic energy should not have the effect of the centrifugal force field in it and is obtained by eliminating all terms involving Ω^2. However, all the nonlinear coupling terms must be retained. Denoting time differentiations by dots, the following kinetic energy expression is obtained after elimination of Ω^2 terms.[4]

$$T = (1/2) \sum_{i=1}^{n} \left[m_i \left\{ \dot{y}_i^2 + 2\dot{y}_i \Omega x_i + 2\Omega \left(y_i \sum_{j=1}^{i-1} l_j \alpha_j \dot{\alpha}_j - (1/2) \dot{y}_i \sum_{j=1}^{i-1} l_j \alpha_j^2 \right) \right\} + J_i \{ \dot{\alpha}_i^2 + 2\Omega \dot{\alpha}_i \} \right] \tag{4}$$

It can be shown[4] that using a one-mode approximation for y_i and α_i leads to an elimination of the problem nonlinearities. With the two-term approximation expressed in Eqs. (1) however, the Lagrangian equations of motion of the rotating Myklestad beam are two nonlinear coupled ordinary differential equations[4] in the normal coordinates $q_1(t)$ and $q_2(t)$. We are interested in a harmonic solution of the form

$$q_1 = B_1 \sin \omega t, \qquad q_2 = B_2 \sin 2\omega t \tag{5}$$

where B_1 and B_2 are arbitrary parameters and ω is the nonlinear frequency. Defining the average Lagrangian $\bar{L}$ as the time integral of the Lagrangian function taken over a period, and using Eqs. (1–5), the Ritz averaging method[2,4] ($\partial \bar{L}/\partial B_1 = 0$ and $\partial \bar{L}/\partial B_2 = 0$) yields the following relationships between the coefficients B_1 and B_2, the frequency ω and the other system parameters, for the fundamental nonlinear mode:

$$B_1^2 = 2L_1L_2/L_3^2, \qquad B_2 = -L_1/L_3 \tag{6}$$

where

$$L_1 = \pi(\omega^2 - \omega_{(1)}^2)M_{(1)}/2\omega, \qquad L_2 = \pi(4\omega^2 - \omega_{(2)}^2)M_{(2)}/2\omega$$

and (7)

$$L_3 = \Omega\pi \sum_{i=1}^{n} \sum_{j=1}^{i-1} m_i l_j C_j^{(1)}(A_i^{(2)}C_j^{(1)} - A_i^{(1)}C_j^{(2)})$$

A two-mode linear analysis using the Ritz variational method gives the natural frequencies of the linearized system as

$$\omega_{L_{(1)}} = \omega_{(1)}, \qquad 2\omega_{L_{(2)}} = \omega_{(2)} \tag{8}$$

where we recognize $\omega_{(1)}$ and $\omega_{(2)}$ as the true linear natural frequencies.

Let us now solve the forced vibration problem with a harmonic force $F \sin \omega_f t$ acting at the beam tip. We consider a steady-state solution of the same form as Eqs. (1) with ω_f replacing ω in Eqs. (5). Two equations can be obtained[4] by the application of a generalized Hamilton's principle which takes into account nonconservative forces which are considered as constants

$$\partial L/\partial B_1 = -\bar{Q}_1, \qquad \partial L/\partial B_2 = -\bar{Q}_2 \tag{9}$$

where $\bar{Q}_1$ and $\bar{Q}_2$ are the average nonconservative generalized forces associated with B_1 and B_2, respectively. Evaluating these generalized forces, using the virtual work concept, and the average Lagrangian in this case, and then performing the required differentiations, Eqs. (9) yield

$$a + br_{a_1} + r_{a_1}^3 = 0, \qquad r_{a_2} = cr_1^2 \tag{10}$$

where the following quantities are defined

$$r_f = \omega_f/\omega_{L_{(1)}}, \qquad r_{a_1} = B_1/B_{1_{st}}, \qquad r_{a_2} = B_2/B_{1_{st}}$$
$$L_{4_f} = FA_n^{(1)}\pi/\omega_f, \qquad a = -(L_{4_f}L_{2_f}/L_3^2B_{1_{st}}^3) \tag{11}$$
$$b = -(2L_{1_f}L_{2_f}/L_3^2B_{1_{st}}^2), \qquad c = -(L_3B_{1_{st}}/2L_{2_f})$$

and

$$B_{1_{st}} = FA_n^{(1)}/\omega_{(1)}^2M_{(1)} \tag{12}$$

is the blade static deflection obtained by putting $\omega_f = 0$ in the first of Eqs. (9). Here, L_{1_f} and L_{2_f} are the L_1 and L_2, respectively, of Eqs. (7) with ω replaced by ω_f, and a, b, and c are functions of the frequency ratio r_f. Equations (10) give us the harmonic response of the blade in its fundamental nonlinear mode.

It can be easily seen that if the first and second normal modes and frequencies are replaced by the pth and $(p+1)$th normal modes and frequencies, respectively, the above analysis generates the pth nonlinear mode. Thus, nonlinear modes other than the fundamental can be readily evaluated.

Numerical Results and Conclusions

The problem of the uniform turbine blade analyzed in Ref. 2 by a continuous system approach was solved by the Myklestad beam formulation presented herein. The blade is 3.62 in. in length and is mounted on a disk of radius 10.38 in. The blade cross-sectional area and moment of inertia are 0.128 in.^{2} and 0.001388 in.^{4}, respectively. The blade material has a density of 0.283 lb/in.^{3} and a Young's modulus of 30×10^6 psi. The amplitude of the exciting force is 500 lb. The blade was represented by seven discrete mass stations and six sections, and shear deformation and rotatory inertia were ignored so as to effect a comparison with the results of Ref. 2. Although it is recognized that the natural frequencies and mode shapes for a Myklestad beam may be obtained by many numerical methods, they are determined in the present investigation by the Myklestad method[3,5-7] using a computer program written for the Sigma-7 digital computer at the University of Texas at Arlington. Numerical results for the fundamental nonlinear mode were obtained for a speed of rotation of 1131 rad/sec with the help of an extension to this program. Figure 2 shows a comparison of the response curves generated with those of Rao and Carnegie.[2] The first and second linear natural frequencies were found to be 6266.75 and 36859.25 rad/sec, respectively. The free vibration results given by Eqs. (6) indicate[4] that the amplitudes become rather large when the frequency is different from the linear natural frequency of the system. The shapes of the response curves plotted are similar to those of Ref. 2. However, one difference noted is that at exciting frequencies less than the linear natural frequency, the second harmonic amplitude is quite small, while according to the analysis of Ref. 2, large amplitudes of the second harmonic can occur at these frequencies. Since the present formulation utilizes the actual normal modes of the linear rotating blade, its results can be considered more dependable than those of Rao and Carnegie,[2] who state that their second harmonic cannot be relied upon with the approximation they have made.

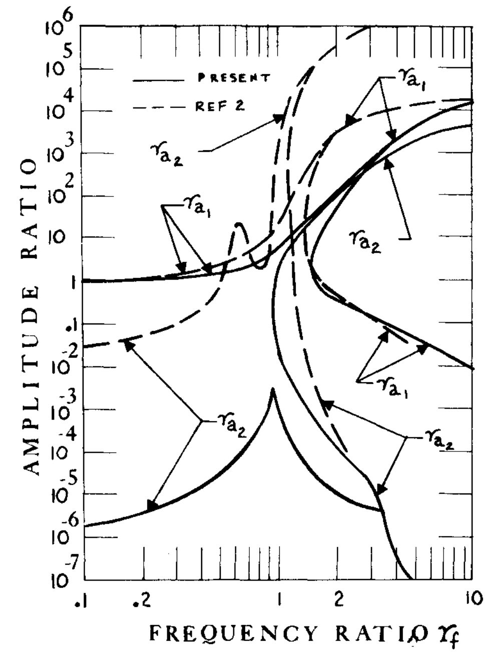

Fig. 2 Blade response in the fundamental nonlinear mode.

The following are the definite advantages of the Myklestad beam representation used in this work over the continuous system formulation employed by Rao and Carnegie[2]: 1) Solutions for nonuniform blades can be easily obtained. 2) Shear deformation and rotatory inertia effects can be taken into account. 3) Nonlinear modes other than the fundamental can be readily evaluated. It is recognized that other methods of representation such as the finite element method can provide very accurate modeling of continuous blades.

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